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**UNC-Charlotte Case Study**

The following project is based on a case study provided to students by Craig DeAlmeida, a Fellow of the Society of Actuaries. The case study walks students through the process of setting product reserves for a Medicare life annuity product. The project is broken up into two sections. The first explains the derivation of concepts used to solve the case study. The second is a step-by-step supplement for future MATH 3129 students to solve the case study. The supplement is broken into segments that correspond to the order in which students learn the material from the “Models for Life Contingencies” exam administered by the Society of Actuaries.

The material for the “Models for Life Contingencies” exam is built upon principles from financial mathematics and probability. The subject matter is used to incorporate mortality risk into the products sold by insurance companies. For a life contingent annuity product, the longer a customer survives, the more the company will have to pay to the customer in the future. The case study seeks to help students apply several concepts from the exam, while also encouraging them to analyze the effects of capital decisions on other business functions.

**Mortality and Survival**

The first and most important subject covered by the exam is mortality. Mortality, or the probability of death, can be observed and calculated for all living creatures. Mortality can also be seen as the probability of failure for inanimate objects, such as lightbulbs or vehicles, or the probability of future events, such as hurricanes or winning the lottery. Primarily, life actuaries use mortality to evaluate the likelihood of receiving premiums or paying benefits in the future.

In standardized practice, we notate mortality as “q” decorated with subscripts. A right subscript indicates the age at the beginning of the period. The left subscript indicates the duration until the end of the period and no left subscript implies a duration of one. For example, tqx is the probability that a person age x dies within t years. The notation for survival, or “p”, follows the same notation principles as mortality. tpx is the complement of tqx and can be calculated as tpx = 1- tqx. Survival rates can also be calculated as the product of other survival rates, or more generally t+spx = tpx\*spx+t.

Another method for calculating survival rates is to use the force of mortality. “Study Manual for SOA Exam MLC Life Contingencies” by Abraham Weishaus defines force of mortality, µx, as “the rate of mortality at age x, given survival to age x.” Mathematically, it can be defined as: µx=, where T0 is the random variable that represents the expected time of death of a person age 0. Combining the topics of survival and force of mortality, we can use the formula; µx+t = - . Thus, to calculate survival, tpx = .

In the case study, the force of mortality increases by 10% every year. To calculate survival rates, we performed the following calculations:

px = = , assuming µ is constant

px+1==()1.1=(px)1.1

Mathematically, this keeps tqx from decreasing below zero as t approaches infinity. This is necessary because tqx represents a probability and therefore must have a value between zero and one, inclusively.

**Illustrative Life Table**

An illustrative life table, commonly referred to as a life table or an actuarial table, is a tool that actuaries can use to visualize survival and calculate actuarial present value. Actuarial present value, like present value in financial mathematics, uses interest rates to consider the time value of money. The difference between the two is that actuarial present value additionally considers the likelihood of the cash flow using mortality. Illustrative life tables include single year mortality and survival rates, cumulative survival rates (tpx, also called remaining cohorts), and expected number of survivors. lx represents the expected number of survivors at age x. lo, or the radix, represents the initial number of lives. The expected number of survivors at any age can be calculated using the formula: lx+t =lx \*tpx. For the case study, we calculated l65 by dividing the sum of contributions of participants age 65, $60 million, by the average contribution per participant, $100 thousand. Current industry mortality tables stop at age 121, so for this project we assume that individuals do not live past that age.

**Life Annuities**

For life contingent annuities, future cash flows are discounted using both interest rates and survival rates. The present value of a certain annuity due, or an annuity paid at the beginning of each period without regard to mortality, is calculated by: =, where n is the number of payments, i is the interest rate, v is (1+i)-1, and d is i/(1+i). The “angle” over the n represents certainty of payment.

Life annuities, by nature, have the added element of randomness. To incorporate randomness, we must introduce the random variable Kx, or curtate life expectancy. Curtate life is the future lifetime of a person without considering the last fractional year of their life. For example, the curtate life of a person dying at age k+m ( where ) would be k. This can also be viewed as people who reach the age k, but die before reaching age k+1. The expected value of curtate life is given as: E(Kx)=k\*k|qx (where k|qx represents deferred mortality and is calculated as, k|qx= tpx\*qx+t.) Using curtate life expectancy, the expected present value of a whole life insurance can be calculated. Whole life insurance on a person age x, denoted as Ax, that pays a benefit, bk, where k is the curtate life of that person, is calculated as: Ax=bk\*k|qx\*v(k+1) = . Life annuity for a person age x is notated as, x, where x represents the age of the participant and does not have an angle over the x because the payments are not guaranteed.

To derive the formula for expected actuarial present value of a life annuity due, begin with the formula for present value of an annuity certain, where the number of periods is equal to the Curtate life expectancy plus one, =. Taking the expected value of both sides of the equation, yields the following: x=. can also be calculated using the formula: bk\*kpx\*vt.

In the case study, each payment at time t of $6000 was discounted to time 0 (age 65) by multiplying it by tp65\*vt. The sum of each of those payments is equal to the expected actuarial present value of the annuity. To calculate the expected present value at any time t, called “reserve” in the case study, we must first calculate the annuity factor for every time period. The annuity factor for any time r is (kpx\*vk)/(rpx\*vr), where 56 is the maximum age less the starting age. The reserve per person at any time t is the annual benefit multiplied by the annuity factor. Multiplying the reserve per person by the number of expected living participants at the time, l65+t, yields the total expected reserve at any time.

**Value at Risk**

Value at Risk (VaR) is a tool that actuaries use to estimate the worst-case scenario for mortality. For a life annuity product, the worst-case scenario for an insurance company would be the improvement of mortality, or decrease in mortality, because they would have to pay out more than they expected. The case study has students solve for the 99% Value at Risk, or the 99th percentile for deaths in the first year and assume that the situation occurs. To calculate the change in reserves using the 99% one-year VaR, students must first calculate the expected change in reserves original mortality assumptions. The expected change in reserves can be calculated as the difference between reserves from time 1 from time 0. It is expected that reserves would decrease over time as participants die and no longer receive benefits. After calculating change in reserves, students must calculate the expected number of deaths in year one. This can be solved as l65-l66.

To calculate the 99% Value at Risk deaths, we must first consider the distribution of deaths in one year. The distribution of deaths in year one is a binomial distribution with parameters n=600 and p=q65. The binomial distribution is a discrete probability distribution with two parameters n and p. The cumulative distribution of a binomial random variable is: . The 99% Value at Risk is the lowest value of x that makes . This value for x can be calculated using the BINOM.INV() function in Exel, which produces the answer 1. Once the VaR is calculated, the difference in expected and 99% VaR deaths can be multiplied by the expected reserve per person. This value represents the unexpected increase in reserves assuming the 99th percentile for deaths in year 1.

**Student Supplement**

Included below is the step-by-step supplement for MATH 3129 students to work through the case study provided by Craig DeAlmeida in Fall 2016. The supplement works through the case study in the order in which students learn the material in class. Relevant excerpts from the case study are included below:

Case Study Excerpt:

“Assume 10 lives per thousand mortality at age 65, um, age last birthday in this case, and assume that the mortality rate by 10% each year, doing the whole converting to a survival rate and taking that to a power to avoid the mortality rate going over 100%.”

Case Study Excerpt:

“The retirement group allocates quotas from each insurance company that participates based on their benefit quotes, and our allocation has been steady at $60 million for several years. The average deposit has also been steady at $100 thousand, so, on average, each participant receives $6000 on their 65th birthday and continues to receive $6000 on subsequent birthdays as long as they live.”

**Case Study Assignment**

1. Using the Excel workbook “Medicare Annuity Case Study”, create an illustrative life table on the tab “1.Life\_Table” as follows:
   1. In cell D5, enter “**=10/1000**” because of the assumption of 10 lives per thousand at age 65. In cell E5, enter “**=1-D5**” because the probability of survivingfrom age 65 to age 66 is the complement of dying between those ages.
   2. We assume that the rate of mortality, μ, increases by 10% every year.

px = = (assuming a constant force of mortality over the year)

px+1 = e-1.1μ = px1.1

Therefore, in cell E6, enter “**=E5^1.1**”. Select cells E6:E61 and use the *Fill Down* command (CTRL + d).

* 1. In cell D6, enter “**=1-E6**” because the probability of dying between ages 66 and 67 is the complement of surviving during that time. Select cells D6:D61 and use the *Fill Down* command.
  2. In cell F5, enter “**1**” because the probability of surviving from age 65 to 65, given that the participant is already 65 is 1. In cell F6, enter “**=E5**” because p65 is equal to 1p65. In cell F7, enter “**=F6\*E6**” because t+sp65=tp65\*sp65+t. Select cells F7:F61 and use the *Fill Down* command.
  3. In cell G5, enter “**=60000000/100000**”. $60 million is from the case study and represents the sum of all participant contributions at age 65. 100 thousand indicates the average deposit per participant. The quotient of the two provides the number of initial participants.
  4. In cell G6, enter “**=G$5\*F6**” because lx+t=lx\*tpx. Select cells G6:G61 and use the *Fill Down* command. The ‘$’ symbol locks the column or row of a particular cell when using ‘Fill’ functions.
  5. Answer the following questions:
     1. What is 15p65?
     2. What is 10q65?
     3. What is p80?
     4. What is 2p80?

1. Using the Excel workbook “Medicare Annuity Case Study”, calculate the expected present value of benefits paid on the tab “2.PV(1Life)” as follows:
   1. To calculate the discount factor for each time period, enter “**=(1.03)^-C5**” in cell G5, because vt=(1+i)-t. Select the range G5:G61 and use the *Fill Down* command.
   2. To find the actuarial present value of each benefit(t) at time 0, enter “**=6000\*F5\*G5**” in cell H5. Select the range H5:H61 and use the *Fill Down* command.
   3. In cell I5, enter “**=SUM(H5:H61)**”. This value represents the expected present value of the benefits paid for one participant.
   4. Answer the following questions:
      1. What is APV(b70)? What is APV(b100)? Looking at the formula for these values, what two factors contribute to the difference in these numbers?
      2. Explain the value of APV(b65). (Hint: What is the annual benefit paid to each living participant? Use the factors from the previous question.)
      3. What is the value of 6000 ?
      4. What is the formula for ?
2. Using the Excel workbook “Medicare Annuity Case Study”, calculate the reserve held by the company at any time. Note, the author of the case study defines the reserve as the expected present value at time t, rather than the difference on the tab “3.Reserve(t)” as follows:
   1. To calculate the discount factor for each period: In cell F5, enter “**=1.03^(-C5)**”. Select cells C5:C61 and use the *Fill Down* command.
   2. The annuity factor for any time r is (kpx\*vk)/(rpx\*vr), where 56 is the maximum age less the starting age. To calculate the annuity factor for each period: In cell G5, enter “**=SUMPRODUCT(E5:E$61,F5:F$61)/(E5\*F5)**”. Select the range G5:G61 and use the *Fill Down* command.
   3. To calculate the reserve per person, multiply the annual benefit by the calculated annuity factor. In cell H5, enter “**=6000\*G5**”. Select the range H5:H61 and use the *Fill Down* command.
   4. To calculate the total reserve at time t, multiply the reserve per person by the expected number of participants alive at that time. In cell J5, enter “**=H5\*J5**”. Select the range J5:J61 and use the *Fill Down* command.
   5. Answer the questions below:
      1. What is the reserve per person at time 0?
      2. What is the reserve per participant at time 1?
      3. What is the reserve per participant at time 56? Explain why.
3. Using the Excel workbook “Medicare Annuity Case Study”, calculate the one year Value at Risk (VaR) at 99% change in reserve for a year of typical sales on the tab “4.VAR” as follows:
   1. To calculate the expected change in reserve without the 99% VaR, enter “**=H4**” in cell I4. In cell I5, enter “**=H5-H4**”.
   2. To calculate the expected deaths in year 1 without the 99% VaR, enter “**=F4-F5**” in cell J5.
   3. The distribution of deaths in year one is a binomial distribution with parameters n=600 and p=q65. The binomial distribution is a discrete probability distribution with two parameters n and p. The cumulative distribution of a binomial random variable is: . The 99% Value at Risk is the lowest value of x that makes . To calculate the number of deaths with Value at Risk at 99% enter “**=BINOM.INV(F4,D4,.01)**” in cell K5.
   4. In cell L5, enter “**=J5-K5**”. This represents the reduction in deaths from expected to 99% VaR.
   5. In cell M5, enter “**=L5\*G5**”. This value indicates the total unexpected increase in reserves based on the one year Value at Risk at 99%.